

# SOME INTERSECTIONS IN THE POINCARÉ BUNDLE AND THE UNIVERSAL THETA DIVISOR ON $\overline{\mathcal{A}}_g$

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**ABSTRACT.** We compute all the top intersection numbers of divisors on the total space of the Poincaré bundle restricted to  $B \times C$  (where  $B$  is an abelian variety, and  $C \subset B$  is any test curve). We use these computations to find the class of the universal theta divisor and  $m$ -theta divisor inside the universal corank 1 semiabelian variety — the boundary of the partial toroidal compactification of the moduli space of abelian varieties. We give two computational examples: we compute the boundary coefficient of the Andreotti-Mayer divisor (computed by Mumford but in a much harder and ad hoc way), and the analog of this for the universal  $m$ -theta divisor.

## 1. INTRODUCTION

The Poincaré bundle is the moduli space of degree 0 line bundles over an abelian variety  $B$ . It is a central object in research of abelian varieties. Yet even the intersection theory on the total space of the Poincaré bundle is not well understood. In this work we consider the restriction of the Poincaré bundle to a test curve  $C \subset B$  times the abelian variety  $B$  itself. We compute the image of the Neron-Severi group of the total space of the Poincaré bundle under this restriction, and the top intersection products of the classes in this group.

The usefulness of this computation lies in the fact that such a test curve  $C$  is one of the two generators for the cone of curves on the partial compactification of the moduli space of principally polarized abelian varieties. Indeed, the universal family of “semiabelian varieties”, which play the same role in the moduli of abelian varieties that the rational nodal curve plays in the moduli of elliptic curves, is naturally the total space of the global Poincaré bundle over the moduli space (see below for a more precise description).

We are thus able to use our computations in the Poincaré bundle to compute the degeneration of the universal theta divisor to the universal semiabelian varieties, with and without level. Finally, the knowledge of the class of the universal semiabelian theta divisor allows us to compute some numerical invariants. We demonstrate this with two computations: a short and straightforward computation of the boundary coefficient of the closure of Andreotti-Mayer divisor (the divisor consisting of principally polarized abelian varieties for which the theta divisor is singular — see [AnMa67]).

This divisor was computed by Mumford in [Mum82] by geometrically interpreting the condition of the semiabelian theta divisor being singular; Yoshikawa in [Yo99] has constructed an explicit modular form defining the Andreotti-Mayer divisor, and thus obtained a formula for its class. The advantage of our approach is that once the intersection-theoretic expression for the divisor on the open part of  $\mathcal{A}_g$  is known, this same expression is used on the universal semiabelian family, with no further geometric intuition required. Using our machinery we are also able to compute the boundary coefficient of a level- $m$  analog of the Andreotti-Mayer divisor: the divisor on the level- $m$  cover of  $\mathcal{A}_g$  consisting of principally polarized abelian varieties for which the  $m$ -theta divisor is singular.

Throughout this paper we work over the complex numbers.

## 2. TOP INTERSECTIONS OF DIVISORS ON THE POINCARÉ BUNDLE

**2.1. Notation** (Moduli space of abelian varieties). In this paper we consider the moduli space of principally polarized complex abelian varieties (ppavs) of dimension  $g$ , which we denote  $\mathcal{A}_g$ . It is a stack, and we will need to be careful with the stackiness, especially when dealing with the level cover and branching along the boundary. However, for the Grothendieck-Riemann-Roch computations that we do stackiness would not be a problem. We denote by  $\mathcal{X}_g \rightarrow \mathcal{A}_g$  the universal family of ppavs, with the fiber over a point  $[A]$  being the abelian variety  $A$  itself.

**2.2. Notation.** Throughout the paper, for an algebraic variety (or a Deligne-Mumford stack)  $X$  we will denote by  $NS(X)$  the Neron-Severi group of numerical equivalence classes of divisors on  $X$ , and by  $CH^*(X)$  the Chow ring of  $X$ .

**2.3. Notation.** For a ppav  $B \in \mathcal{A}_g$  we denote by  $\Theta_B \subset B$  the (symmetric) divisor of its principal polarization. For  $m \in \mathbb{Z}$  we denote by  $m_B : B \rightarrow B$  the multiplication by  $m$  map,  $m_B(z) := mz$  on  $B$ . This map will turn out to be important for level cover considerations. Finally for any point  $b \in B$  we denote by  $\tau_b : B \rightarrow B$  the translation by  $b$  map  $\tau_b(z) := z + b$ .

**The Neron-Severi group  $NS(B \times B)$  and its restriction to  $NS(B \times C)$ .**

**2.4. Notation.** Let  $B \in \mathcal{A}_{g-1}$  be a very general ppav, and let  $C \subset B$  be a very general curve of degree  $n$  in it, i.e. such that  $C \cdot \Theta_B = n$ . In the following text we denote by  $r$  the map  $C \rightarrow B$ , and consider the image of the restriction map

$$r^* : NS(B \times B) \rightarrow NS(B \times C).$$

Let  $\mathcal{P}$  be the Poincaré bundle over  $B \times B$  — it is the universal degree 0 line bundle over  $B$ , i.e. the unique line bundle such that  $\mathcal{P}|_{0 \times B}$  is trivial, and  $\mathcal{P}|_{B \times b}$  for any  $b \in B$  is the degree zero line bundle on  $B$  corresponding to  $b$ , when we identify  $B$  with the dual abelian variety  $B^\vee = \text{Pic}^0(B)$  by using the principal polarization on  $B$ .

Let us denote by  $\alpha = c_1(\mathcal{P})$  the first Chern class of the Poincaré bundle on  $B \times B$  (restricted to  $B \times C$ ). Let  $\mu, \eta$  be the restrictions of  $\Theta_B \times B$  and  $B \times \Theta_B$  to  $B \times C$ . We also define three curves in  $B \times C$ :

$$\mu^* := \{(x, 0) | x \in C\}, \quad \eta^* := \{(0, x) | x \in C\}, \quad \delta^* := \{(x, x) | x \in C\}.$$

Finally denote by  $s$  the “shift” automorphism

$$s : B \times B \rightarrow B \times B$$

$$(z, b) \mapsto (z + b, b),$$

and note that  $s$  restricts to an automorphism of  $B \times C$ . We then denote by  $s^*$  the action of  $s$  on  $NS(B \times B)$  by pullback.

**2.5. Proposition.** *We have the following intersection numbers on  $B \times C$ :*

curve \ divisor	$\mu$	$\eta$	$\alpha$
$\mu^*$	$n$	0	0
$\eta^*$	0	$n$	0
$\delta^*$	$n$	$n$	$2n$

*Proof.* Computing the intersection of the classes  $\mu$  and  $\eta$  with these curves is easy: we just forget the irrelevant factor. To compute the intersection of the class  $\alpha$  of the Poincaré bundle with these curves note that the Poincaré bundle is trivial on  $0 \times C$  and  $B \times 0$ , so that the intersections with  $\mu^*$  and  $\eta^*$  are zero, while the restriction of the Poincaré bundle to the diagonal, pulled back to one of the factors, is  $\mathcal{O}(2\Theta_B)$  (see [Mum82], Second statement of Proposition 1.8), the degree of which on  $C$  is  $2n$ . Thus we are done.  $\square$

**2.6. Corollary.** *The group  $r^*(NS_{\mathbb{Q}}(B \times B)) \subset NS_{\mathbb{Q}}(B \times C)$  is generated by  $\alpha, \mu, \eta$ .*

*Proof.* By [BiLa04]  $NS(B \times B)$  is 3-dimensional, since  $B$  is very general and does not have automorphisms, but we have shown that  $\alpha, \mu$ , and  $\eta$  are linearly independent, since their intersections with three curves are linearly independent.  $\square$

**2.7. Proposition-Definition.** For  $N \in \mathbb{Z}$  let  $\mu_N := (s^*)^N(\mu)$ , then in  $NS(B \times C)$  we have  $\mu_N \equiv \mu + N\alpha + N^2\eta$ .

*Proof.* Note that

$$\begin{aligned} s^N(\mu^*) &= \{(x, 0) | x \in C\} = \mu^*, \quad s^N(\eta^*) = \{(Nx, x) | x \in C\}, \\ s^N(\delta^*) &= \{(N+1)x, x) | x \in C\}. \end{aligned}$$

We recall that if  $E \in CH^e(B)$ , then  $[(m_B)_*E] = m^{2e}E$  (using Poincaré duality and applying Theorem 6.2 from [Mi98]). Hence

$$\begin{aligned} \mu_N \cdot \mu^* &= \mu \cdot s^N(\mu^*) = \mu \cdot \mu^* = n, \\ \mu_N \cdot \eta^* &= \mu \cdot s^N(\eta^*) = \langle \Theta_B \cdot \{Nx | x \in C\} \rangle_B = N^2 \langle \Theta_B \cdot C \rangle_B = N^2 n \\ \mu_N \cdot \delta^* &= \mu \cdot s^N(\delta^*) = \langle \Theta_B \cdot \{(N+1)x | x \in C\} \rangle_B \\ &= (N+1)^2 \langle \Theta \cdot C \rangle_B = (N+1)^2 n. \end{aligned}$$

The result now follows from the intersection matrix in Proposition 2.5.  $\square$

**2.8. Corollary.** *The action of  $s^*$  on  $r^*(NS(B \times B))$  is given by*

$$s^*(\mu) = \mu + \alpha + \eta; \quad s^*(\alpha) = \alpha + 2\eta; \quad s^*(\eta) = \eta.$$

*Proof.* By the previous Proposition we have for all  $N$ :

$$\begin{aligned} s^*(\mu + N\alpha + N^2\eta) &= s^*\mu_N = \mu_{N+1} = \mu + (N+1)\alpha + (N+1)^2\eta \\ &= (\mu + \alpha + \eta) + N(\alpha + 2\eta) + N^2\eta. \end{aligned}$$

The result follows by equating the coefficients of the corresponding powers of  $N$  on both sides.  $\square$

### Top intersections in the Chow ring of $B \times C$ .

**2.9. Proposition.** *The top intersections numbers of divisors in  $r^*(NS(B \times B))$  on  $B \times C$  are completely determined by the following relations:*

$$\begin{aligned} (\square) \quad \eta^2 &= 0 & (\diamondsuit) \quad \eta\mu^{g-1} &= n(g-1)! \\ (\triangle) \quad \alpha\eta &= 0 & (\heartsuit) \quad \alpha^k\mu^{g-k} &= \begin{cases} -2(g-2)! & \text{if } k = 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

*Proof.* equation  $\square$  follows from the fact that  $\eta$  is a pullback of a class on a curve.

For equation  $\triangle$ , note that the Poincaré bundle is numerically trivial on all horizontal and vertical fibers of  $B \times B$ . Since  $\alpha \cdot \eta$  is the  $c_1$  of the restriction of the Poincaré bundle to  $\eta$ , which is geometrically just  $B$  times  $n$  points, which is zero.

To prove  $\diamondsuit$ , we compute

$$\eta\mu^{g-1} = \mu^{g-1}|_{\eta} = \mu^{g-1}|_{B \times \{n \text{ points}\}} = n\Theta_B^{g-1}|_B = n(g-1)!,$$

since restricted to  $B \times pt$ , the class  $\mu$  is just  $\Theta_B$ .

To prove equation  $\heartsuit$  note that the class  $\mu_N$  is a pullback of  $\Theta_B$  from  $B$  to  $B \times C$  under the map  $(z, b) \rightarrow (z + Nb)$ . Thus, as a pullback class from a  $(g-1)$ -dimensional variety, its  $g$ 'th power is zero:

$$\begin{aligned} 0 &= \mu_N^g = ((\mu + N\alpha) + N^2\eta)^g \stackrel{\square}{=} (\mu + N\alpha)^g + N^2g\eta(\mu + N\alpha)^{g-1} \\ &\stackrel{\triangle}{=} (\mu + N\alpha)^g + N^2g\eta\mu^{g-1} \stackrel{\diamondsuit}{=} (\mu + N\alpha)^g + N^2gn(g-1)!. \end{aligned}$$

This equality must hold for all  $N$ , which implies that in the binomial expansion of  $(\mu + N\alpha)^g$  only the term quadratic in  $N$  can be non-zero. Since the term quadratic in  $N$  is  $N^2g(g-1)\alpha^2\mu^{g-2}/2$ , the equality  $\heartsuit$  follows.  $\square$

### Top intersections in the Chow ring of the total space of the Poincaré bundle.

**2.10. Notation.** Let  $E$  be the trivial line bundle on  $B \times B$ . Denote by  $\tilde{\mathcal{P}} \rightarrow B \times C$  the restriction of the bundle  $\mathbb{P}(E \oplus \mathcal{P})$  from  $B \times B$  to  $B \times C$ , and set  $\xi := c_1(\mathcal{O}_{\tilde{\mathcal{P}}}(1))$ . We define also the  $0, \infty$  section of this bundle by setting:

$$\mathcal{P}_0 := \mathbb{P}(E \oplus \{0\})|_{B \times C} \quad \mathcal{P}_\infty := \mathbb{P}(\{0\} \oplus \mathcal{P})|_{B \times C}.$$

We denote the pullback of any class  $F \in CH^*(B \times C)$  to  $CH^*(\tilde{\mathcal{P}})$  by  $\tilde{F}$ .

**2.11. Proposition.** *We list some standard properties of vector bundles in our context:*

- (1) *The class of the vertical fiber  $\mathbb{P}^1$  of  $\tilde{\mathcal{P}} \rightarrow B \times C$  has intersection 1 with  $\xi$ , and 0 with pullbacks of all the divisors from  $B \times C$ .*
- (2) *The projection  $\mathcal{P}_0 \rightarrow \mathbb{P}(E) \rightarrow B \times C$  is an isomorphism.*
- (3) *In the Neron-Severi group  $NS(\tilde{\mathcal{P}})$  we have  $\mathcal{P}_0 - \mathcal{P}_\infty \equiv \widetilde{c_1(\mathcal{P})} - \widetilde{c_1(E)} = \tilde{\alpha}$*
- (4) *In the Neron-Severi group  $NS(\tilde{\mathcal{P}})$  we have  $\mathcal{P}_\infty \equiv \xi$ .*

**2.12. Proposition.** *The Chow ring of  $\tilde{\mathcal{P}}$  is given by*

$$CH^*(\tilde{\mathcal{P}}) = CH^*(B \times C)[\xi]/(\xi^2 + \tilde{\alpha}\xi).$$

*Proof.* Since  $\tilde{\mathcal{P}} = \mathbb{P}(E \oplus \mathcal{P})$  we have ([Fu98], remark 3.2.4)

$$0 = \xi^2 + c_1(E \oplus \mathcal{P})\xi + c_2(E \oplus \mathcal{P}) = \xi^2 + \tilde{\alpha}\xi.$$

□

**2.13. Remark.** Note that this matches with the obvious fact that the 0 and infinity sections do not intersect: indeed, we have  $\mathcal{P}_0 = \mathcal{P}_\infty + \tilde{\alpha} = \xi + \tilde{\alpha}$ , and so it checks out that

$$\mathcal{P}_0 \mathcal{P}_\infty = (\xi + \tilde{\alpha})\xi = 0.$$

**2.14. Proposition.** *The top intersection products of divisors on  $\tilde{\mathcal{P}}$  are completely determined by the pullback of the relations  $\square, \triangle, \diamond, \heartsuit$  together with:*

(■)  $\xi^2 = -\tilde{\alpha}\xi$ ,      (▲) *the map  $CH^*(B \times C) \rightarrow CH^*(\xi)$  arising from statements 2,4 in Prop. 2.11 is an isomorphism,*

*and the fact that top intersections of pullback classes are 0 (for dimension reasons).*

**2.15. Remark** (Motivation for the notations). We only consider intersections of two divisors or top intersections. The identities for top intersection numbers are denoted by card suits, while the equations for the squares of the generators are denoted by squares. The triangles thus denote relations for products of two different classes. Moreover, the pullback relations are white, while the relations involving the fiber generator  $\xi$  are black.

This means that in a computation of a top intersection number on  $\tilde{\mathcal{P}}$  we would typically do the following: first apply black relations (and the fact that the top intersection of pullback classes is zero) to reduce the computation

to a computation on  $B \times C$ , then apply  $\square$  and  $\triangle$  to get rid of redundant intersections, and finally the card suit relations to get actual numbers.

**2.16. Remark** (Computational Trick). We conclude this section with a small computational trick, coming from the relations we already deduced:

$$\begin{aligned}
(\xi + a\tilde{\mu} + b\tilde{\alpha})^{g+1} &= \sum_{k=0}^{g+1} \binom{g+1}{k} \xi^k (a\tilde{\mu} + b\tilde{\alpha})^{g+1-k} \\
&\stackrel{\blacksquare}{=} \sum_{k=1}^{g+1} \binom{g+1}{k} \xi(-\tilde{\alpha})^{k-1} (a\tilde{\mu} + b\tilde{\alpha})^{g+1-k} \stackrel{\blacktriangle}{=} \sum_{k=1}^{g+1} \binom{g+1}{k} (-\alpha)^{k-1} (a\mu + b\alpha)^{g+1-k} \\
&\stackrel{\heartsuit}{=} \sum_{k=1}^3 \binom{g+1}{k} (-\alpha)^{k-1} \binom{g+1-k}{3-k} (b\alpha)^{3-k} (a\mu)^{g-2} \\
&= \binom{g+1}{3} (a\mu)^{g-2} \alpha^2 (1 - 3b + 3b^2) \\
&\stackrel{\heartsuit}{=} -n \binom{g+1}{3} a^{g-2} 2(g-2)! (b^3 - (b-1)^3) = -\frac{n(g+1)!}{3} a^{g-2} (b^3 - (b-1)^3).
\end{aligned}$$

We will denote this relation in the sequel by “ $T$ ” (for “trick”).

### 3. PPAVS AND RANK ONE DEGENERATIONS

#### Rank 1 degenerations of abelian varieties, and moduli.

**3.1. Definition** (Semiabelian varieties). A (non-normal compactification of a rank-one-degenerated principally polarized complex) semiabelian variety is constructed as follows. Let  $B \in \mathcal{A}_{g-1}$ ,  $b \in B$ , and let  $S \rightarrow B$  be the line bundle corresponding to  $b$  under the identification  $B \cong \text{Pic}^0 B$ . Complete  $S$  fiberwise to a  $\mathbb{P}^1$ -bundle  $\tilde{S} \rightarrow B$ , and then identify the 0 and  $\infty$  sections of  $\tilde{S}$  (each a copy of  $B$ ) with a shift by  $b$ , i.e. define  $\overline{S} := \tilde{S}/(z, 0) \sim (\tau_b z, \infty)$  (recall that we defined  $\tau_b$  to be the map  $z \mapsto z + b$  on  $B$ ). The divisor of the principal polarization  $\Theta \subset \overline{S}$  is a section  $(B \setminus (\Theta_B \cap \tau_b \Theta_B)) \rightarrow \tilde{S}$ , together with the entire fibers over  $\Theta_B \cap \tau_b \Theta_B$ , such that it intersects the 0-section of  $\tilde{S}$  in  $\Theta_B$ , and correspondingly the  $\infty$ -section of  $\tilde{S}$  in  $\tau_b \Theta_B$ , thus giving a well-defined divisor on  $\overline{S}$ . Notice that the compactification  $\overline{S}$  no longer admits a projection map to  $B$ . We also note that if we started with the point  $-b$  instead of  $b$ , the resulting object is going to be isomorphic — indeed, any semiabelian variety has an involution interchanging the 0 and  $\infty$  sections, and this involution sends a semiabelian variety with bundle  $b$  on the base (the 0 section) to the one with bundle  $-b$ . This means that the family of semiabelian varieties with base  $B$  is parameterized by  $B/\pm 1$ , and thus this family is singular at points of order two on  $B$ .

**3.2. Definition** (Partial compactification). We denote  $\mathcal{A}_g^1$  the partial compactification of  $\mathcal{A}_g$  obtained by “adding” the locus of rank 1 semiabelian

varieties: See [Al02] (setup 1.2.8), [Ol05] for the construction of the second Voronoi toroidal compactification  $\overline{\mathcal{A}}_g$  of  $\mathcal{A}_g$ , over which there exists a universal family  $\overline{\mathcal{X}}_g$ . The partial compactification  $\mathcal{A}_g^1$  is a subscheme of  $\overline{\mathcal{A}}_g$ .

The locus of rank 1 semiabelian varieties in  $\mathcal{A}_g^1$  forms the boundary divisor denoted  $\Delta \subset \mathcal{A}_g^1$ . We denote by  $\mathcal{X}_g^1$  the partial compactification of the universal family of abelian varieties, which is an extension of the family  $\mathcal{X}_g \rightarrow \mathcal{A}_g$  such that the fibers over the boundary are semiabelian varieties. We denote by  $\mathcal{D} \subset \mathcal{X}_g^1$  the universal divisor of the principal polarization, i.e. the universal theta divisor.

**3.3. Definition** (The universal rank 1 degeneration — see [Mum82] 1.8). The universal family of semiabelian varieties in  $\partial \mathcal{X}_g^1$  lying over  $[B] \in \mathcal{A}_{g-1}$ , i.e. the fiber over  $[B]$  of the map  $\mathcal{X}_g^1 \rightarrow \mathcal{A}_g^1 \rightarrow \mathcal{A}_{g-1}$ , is defined in the following way: consider the  $\mathbb{P}^1$ -bundle  $\tilde{\mathcal{P}} = \mathbb{P}(E \oplus \mathcal{P})$  over  $B \times B$ . Denote by  $\mathcal{P}_0$  and  $\mathcal{P}_\infty$  the zero and infinity sections of  $\tilde{\mathcal{P}}$  over  $B \times B$ , respectively. Then the universal semiabelian variety over  $B \times B$  (without level structure) is obtained from  $\tilde{\mathcal{P}}$  by identifying  $\mathcal{P}_0$  and  $\mathcal{P}_\infty$  with a shift by  $b$ :

$$\overline{\mathcal{P}} := \tilde{\mathcal{P}}/(z, b, 0) \sim (s(z, b), \infty),$$

and further taking the quotient under  $b \mapsto -b$ .

From now on we will restrict this construction to our test curves, and use the same letters to denote the restrictions of the glued bundle and its normalization from  $B \times B$  to  $B \times C$ . Since we chose the curve  $C \subset B$ , we can make sure that  $C \cap (-C) = \emptyset$ , and thus in working with  $B \times C$  we can ignore the sign involution of  $b$ .

**3.4. Remark.** It is known since Tai's work in [Ta82] that one can construct some  $\mathcal{A}'_g \supset \mathcal{A}_g^1$  such that  $\mathcal{A}'_g \setminus \mathcal{A}_g^1$  has codimension 2 in  $\mathcal{A}'_g$ , and  $\mathcal{A}'_g$  has only canonical singularities (the rigorous proof of this is given in Shepherd-Barron's [S-B05] and its addendum, to appear).

#### The universal theta divisor on rank 1 degenerations.

**3.5. Proposition.** *The top intersection number of the universal theta divisor over  $B \times C$  restricted to  $\mathcal{P}_0$  is 0.*

*Proof.* By definition on one semiabelian variety  $\tilde{S} \rightarrow B$  the restriction of the theta divisor to the 0-section  $B = S_0 \subset \tilde{S}$  is just  $\Theta_B$ , and thus  $\mathcal{D}|_{\mathcal{P}_0} = p^* \Theta_B$ , where  $p : B \times C \rightarrow B$  is the projection. Hence the top intersection is a pullback of a top+1 intersection product on  $B$ .  $\square$

**3.6. Theorem.** *The class of the universal theta divisor on the normalization  $\tilde{\mathcal{P}}$  of the universal semiabelian variety, restricted to  $B \times C$ , is equal to*

$$\mathcal{D} = \xi + \tilde{\mu} + \frac{1}{2}\tilde{\alpha} + \frac{1}{4}\tilde{\eta}$$

*in  $r^*(NS(\tilde{\mathcal{P}}))$ .*

**3.7. Remark.** Below we give an elementary algebraic derivation of this formula from the geometric description of the universal semiabelian family. Alternatively one could work out explicitly the degenerations of the theta function along the boundary — this approach is followed in [HuWe] for the case of  $g = 2$ , and one easily computes the class of the universal semiabelian theta function given there to be what we claim. Our approach has the advantage of giving a way to straightforwardly deal with the computation on the level cover as well.

*Proof.* Denote the coefficients

$$\mathcal{D} = c_\xi \xi + c_\mu \tilde{\mu} + c_\alpha \tilde{\alpha} + c_\eta \tilde{\eta}.$$

We first compute  $c_\xi$ ; this coefficient can be computed after intersecting with  $\eta$ , i.e. on the fiber over one moduli point in  $\partial \mathcal{A}_g$ . We compute it by considering two families in  $\mathcal{A}_g^1$  with the same flat limit, which is the trivial bundle over  $B$  with 0 and  $\infty$  sections identified by identity. Our first family is the family of semiabelian varieties parameterized by  $(B, b)$  where  $b \rightarrow 0$ . Our second family is the family  $B \times E$  where  $E$  is a moving elliptic curve that degenerates to the rational nodal curve. Since the theta divisor on the second family is numerically given by  $\Theta_B \times E + B \times pt$ , the limit theta divisor is given by  $\Theta_B \times (\mathbb{P}^1/0 \sim \infty) + B \times pt$ . Since the intersection of this limit theta divisor with a general fiber of the bundle is 1 (coming from the  $B \times pt$  component), the same holds in the first family.

We note that with a little more work we could have recovered the coefficient of  $\mu$ , but we will get it almost for free below.

Next we compute the coefficients  $c_\mu, c_\alpha$ : note that by (■)

$$\mathcal{D}|_{\mathcal{P}_\infty} = (\xi + c_\mu \tilde{\mu} + c_\alpha \tilde{\alpha} + c_\eta \tilde{\eta})|_\xi = (c_\mu \tilde{\mu} + (c_\alpha - 1) \tilde{\alpha} + c_\eta \tilde{\eta})|_{\mathcal{P}_\infty}$$

and

$$\mathcal{D}|_{\mathcal{P}_0} = (\xi + c_\mu \tilde{\mu} + c_\alpha \tilde{\alpha} + c_\eta \tilde{\eta})|_{\xi + \mathcal{P}} = (c_\mu \tilde{\mu} + c_\alpha \tilde{\alpha} + c_\eta \tilde{\eta})|_{\mathcal{P}_0}.$$

Since the theta divisor is a stable limit of smooth connected varieties, it is connected on codimension 1. This means that if we identify both  $\mathcal{P}_0$  and  $\mathcal{P}_\infty$  with  $B \times C$ , then  $\mathcal{D}|_{\mathcal{P}_0}$  is the pullback of  $\mathcal{D}|_{\mathcal{P}_\infty}$  under the shift  $s$ , i.e.

$$\begin{aligned} c_\mu \mu + c_\alpha \alpha + c_\eta \eta &= s^*(c_\mu \mu + (c_\alpha - 1) \alpha + c_\eta \eta) \\ &= c_\mu(\mu + \alpha + \eta) + (c_\alpha - 1)(\alpha + 2\eta) + c_\eta \eta \\ &= c_\mu \mu + (c_\mu + c_\alpha - 1) \alpha + (c_\mu + 2c_\alpha + c_\eta - 2) \eta. \end{aligned}$$

Comparing the  $\alpha$  coefficient we have  $c_\mu = 1$ , and comparing  $\eta$  coefficients we get

$$c_\alpha = (2 - c_\mu)/2 = 1/2.$$

To compute  $c_\eta$ , we observe that by Proposition 3.5 we have

$$\begin{aligned} 0 &= (\mathcal{D}|_{\mathcal{P}_\infty})^g \stackrel{\blacksquare, \blacktriangle}{=} (\mu - \frac{1}{2}\alpha + c_\eta \eta)^g \stackrel{\square, \triangle}{=} gc_\eta \eta \mu^{g-1} + (\mu - \frac{1}{2}\alpha)^g \\ &\stackrel{\heartsuit, \diamond}{=} gc_\eta (g-1)! + \binom{g}{2} (-2(g-2)!) \frac{1}{4} = g! \left( c_\eta - \frac{1}{4} \right) \end{aligned}$$

□

3.8. In [Mum82] Mumford computed the class in  $NS(\mathcal{A}_g^1)$  of the closure in  $\mathcal{A}_g^1$  of the Andreotti-Mayer divisor. Since  $NS(\mathcal{A}_g^1)$  is spanned by the Hodge class and the boundary class, he had to compute these two coefficients.

Mumford's computation of the Hodge coefficient of the class of the Andreotti-Mayer divisor (in [Mum82] Proposition 3.1) uses the ramification formula (see [Fu98] Example 9.3.12):

$$\mathbb{R}(f) = (c(f^*(T_C))c(T_{\mathcal{D}})^{-1})_g,$$

where  $f : \mathcal{D} \rightarrow C$  is the universal theta divisor over a smooth curve in  $\mathcal{A}_g$ . In practice, Mumford breaks the morphism  $f$  to a composition  $\mathcal{D} \xrightarrow{i} \mathcal{X} \rightarrow C$ , where  $\mathcal{X}$  is the universal abelian variety over  $C$ , and gets (we omit obvious pullback notations):

$$\begin{aligned} c(T_{\mathcal{D}}) &= c(T_{\mathcal{D}/C})c(T_C) \Rightarrow \mathbb{R}(f) = (c(T_{\mathcal{D}/C})^{-1})_g, \\ c(T_{\mathcal{D}/C}) &= c(T_{\mathcal{D}/\mathcal{X}})c(T_{\mathcal{X}/C}) = (1 - \mathcal{D})c(T_{\mathcal{X}/C}) \Rightarrow \\ i_*\mathbb{R}(f) &= (c(\Omega_{\mathcal{X}/C})(1 - \mathcal{D})^{-1})_g \cdot D = \sum_{i=0}^g c_i(\Omega_{\mathcal{X}/C})\mathcal{D}^{g+1-i}. \end{aligned}$$

Mumford proceeds to compute the class — see the second part of Proposition 3.1 in [Mum82].

To compute the boundary coefficient, Mumford uses two hard ad-hoc arguments. However, armed with our understanding of the theta divisor we can give an alternative straightforward computation:

3.9. **Proposition.** *Let  $C \subset B$  be a test curve as above, let  $f : \overline{\mathcal{D}} \rightarrow C$  be the universal semiabelian theta divisor over the test curve  $C$  (so that its lifting to  $\tilde{\mathcal{P}}$  from  $\overline{\mathcal{P}}$  is  $\mathcal{D}$ ); then  $\mathbb{R}(f) = \frac{n(g+1)!}{6}$ .*

*Proof.* Since the divisor  $\mathcal{D}$  is singular we cannot use the “smooth” ramification formula. Instead we use the “singular” version:

$$\mathbb{R}(f) = (c(f^*(T_C))s(\Delta_{\overline{\mathcal{D}}}, \overline{\mathcal{D}} \times \overline{\mathcal{D}}))_g.$$

See [Fu98] the end of example 9.3.12 for details (compare Johnson/Fulton-Laksov singular double point formula in 9.3.6 to the double point formula in Theorem 9.3).

We compute this intersection product by pulling back under  $\pi : \mathcal{D} \rightarrow \overline{\mathcal{D}}$ . By the definition of the Segre class we have:

$$\begin{aligned} \pi^*\mathbb{R}(f) &= (c(\pi^*f^*(C))s((\pi \times \pi)^*\Delta_{\overline{\mathcal{D}}}, \mathcal{D} \times \mathcal{D}))_{2g-1} \\ &= (c(\pi^*f^*(C))c(T_{\mathcal{D}/\overline{\mathcal{D}}})c(\mathcal{D})^{-1})_g. \end{aligned}$$

By definition we have  $c(T_{\mathcal{D}/\overline{\mathcal{D}}}) = (\mathcal{P}_0 + \mathcal{P}_\infty)|_{\mathcal{D}} = (2\xi + \tilde{\alpha})|_{\mathcal{D}}$ . However, by example [Fu98] 3.2.11, the total Chern class of the relative tangent bundle

is

$$\begin{aligned}
c_t(T_{\tilde{\mathcal{P}}/C}) &= (1 + \xi t)^2 c_{\frac{t}{1+\xi t}}(E \oplus \mathcal{P}) \\
&= (1 + \xi t)^2 c_{t(1-\xi t)}(\mathcal{P}) = (1 + 2\xi t + \xi^2 t^2)(1 + \tilde{\alpha} t(1 - \xi t)) \\
&= 1 + (2\xi + \tilde{\alpha})t + (\xi^2 - 2\tilde{\alpha}\xi + 2\xi\tilde{\alpha})t^2 \triangleq 1 + (2\xi + \tilde{\alpha})t.
\end{aligned}$$

Hence, factoring the map  $\mathcal{D} \rightarrow C$  through  $\mathcal{D} \xrightarrow{i} \tilde{\mathcal{P}} \rightarrow C$ , similarly to the case of smooth abelian varieties, we have

$$i_* \pi^* \mathbb{R}(f) = \left[ \sum c_i (1 + 2\xi + \tilde{\alpha} - (2\xi + \tilde{\alpha})) D^{g-i} \right]_{\deg g} \cdot \mathcal{D}$$

Hence we have:

$$\begin{aligned}
i_* \pi^* \mathbb{R}(f) &= \mathcal{D}^{g+1} = \left( \xi + \tilde{\mu} + \frac{1}{2}\tilde{\alpha} + \frac{1}{4}\tilde{\eta} \right)^{g+1} \\
&= (g+1) \frac{\tilde{\eta}}{4} \left( \xi + \tilde{\mu} + \frac{\tilde{\alpha}}{2} \right)^g + \left( \xi + \tilde{\mu} + \frac{\tilde{\alpha}}{2} \right)^{g+1} \\
&\stackrel{T}{=} (g+1) \frac{\tilde{\eta}}{4} g \xi \tilde{\mu}^{g-1} + \frac{n(g+1)!}{3} \left( \left( \frac{1}{2} \right)^3 - \left( \frac{-1}{2} \right)^3 \right) \\
&= n(g+1)! \left( \frac{1}{4} - \frac{1}{12} \right) = \frac{n(g+1)!}{6}.
\end{aligned}$$

□

#### 4. GEOMETRY OF THE LEVEL COVER $\mathcal{A}_g(m)$ OF $\mathcal{A}_g$

**Background: level structure on rank 1 degenerations of abelian varieties.**

**4.1. Definition** ( $m$ -torsion points and level moduli space). Given a ppav  $A \in \mathcal{A}_g$  and  $m \in \mathbb{N}$ , we denote by  $A[m]$  the set of points of order  $m$  in  $A$ ; note that since  $m$  is prime to the characteristic of the base field, which we assumed to be 0, we have  $A[m] \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ .

We denote by  $\mathcal{A}_g(m)$  the moduli space of ppavs together with a choice of a symplectic basis for the set of  $m$ -torsion points, up to isomorphisms. Here symplectic means with respect to the Weil pairing on  $A[m]$  — see [Mi98] I.12 for a complete definition. The precise definition of the pairing would not matter for us, and thus we do not give it.

The forgetful map  $\mathcal{A}_g(m) \rightarrow \mathcal{A}_g$  is a finite cover with fiber over  $[A] \in \mathcal{A}_g$  being the set of all symplectic bases for  $A[m] \cong (\mathbb{Z}/m\mathbb{Z})^{2g}$ , and with the deck group  $\mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$ .

The fine moduli stack  $\mathcal{A}_g(m)$  carries a universal family  $\mathcal{X}_g(m) \rightarrow \mathcal{A}_g(m)$ , where the forgetful map lifts to the multiplication by  $m$  map  $\left[ A, \xrightarrow{\text{basis}} \text{for } A[m] \right] \xrightarrow{m_A} A/A[m]$  over every moduli point  $[A] \in \mathcal{A}_g$ .

**4.2. Definition** (Global theta divisors). We define the universal order  $m$  theta divisor on  $\mathcal{X}_g(m)$  to be the pullback of the universal theta divisor  $\mathcal{D}$  under the map

$$q : \mathcal{X}_g(m) \longrightarrow \mathcal{X}_g$$

$$[A, \{a_1, \dots, a_g, a'_1, \dots, a'_g\}] \xrightarrow{id} [A/\text{span}(a_1, \dots, a_g)],$$

where  $a_1, \dots, a_g, a'_1, \dots, a'_g$  denotes a symplectic basis of  $A[m]$ .

**4.3.** Below we explain in what sense we have a level structure on a semiabelian variety. We do this by analyzing the limit of the map  $q : \mathcal{X}_g(m) \rightarrow \mathcal{X}_g$  defined above. We now fix (up until Proposition 4.8) a moduli point  $[B] \in \mathcal{A}_{g-1} \subset \mathcal{A}_g^{\text{Sat}}$ , and consider the limit of the map  $q$  over the preimage of  $[B] \in \partial \mathcal{A}_g^1 = \mathcal{X}_{g-1}/\pm 1$ .

**4.4. Definition** (Level  $m$  rank-1 semiabelian varieties). The definition below is the rank 1 case of the more general definition given in [Al02] set-up 1.2.8.

Let  $b$  be a non zero point in  $B$ , and let  $S$  be the degree 0 line bundle over  $B$  associated to  $b$ . Let then  $\tilde{S}^{(i)} := \mathbb{P}(E \oplus S)$ , for  $i = 0, \dots, m-1$ , be  $m$  copies of the projectivization of this bundle, let  $S^{(i)}$  be  $\tilde{S}^{(i)}$  minus the 0 and  $\infty$  sections, and let  $\tilde{S}(m) := \sqcup \lim_{i=0}^{m-1} \tilde{S}_i$  be the disjoint union of  $m$  copies. The (non-normal compactified rank 1) level  $m$  semiabelian variety is

$$\overline{S}(m) = \tilde{S}(m)/(z, 0)^{(i+1)} \sim (z, \infty)^{(i)} \text{ and } (z, 0)^0 \sim (z + b, \infty)^{(m-1)}.$$

See example 4a in page 40 of [Hu00b] for a “graphical” description.

We denote by  $\mathcal{A}_g(m)^1$  the partial compactification of  $\mathcal{A}_g(m)$  obtained by “adding” the locus  $\Delta(m)$  of level  $m$  rank 1 semiabelian varieties, in the same sense of Definition 3.2: By [Al02] setup 1.2.8,  $\mathcal{A}_g(m)^1$  is the underlying scheme of a substack of projective fine moduli stack: the second Voronoi compactification of  $\mathcal{A}_g(m)$ . The fibers of the universal family  $\mathcal{X}_g(m)^1$  over  $\mathcal{A}_g(m)^1$  are as defined above.

**4.5.** We note that no claim was made (yet) on how the level semiabelian varieties form a universal family, i.e. on how the semiabelian varieties with  $B$  and  $b$  varying fit together in a family; this follows below.

**4.6 (Local monodromy).** Let  $\pi : U \rightarrow M$  be a fine moduli stack, and let  $\gamma$  be an automorphism of  $M$  with a fixed point  $p$ . Then  $\gamma$  induces an automorphism of the fiber over  $p$ .

**4.7. Lemma.** *Let  $[B, b] \in \partial \mathcal{A}_g(m)$  be a boundary point, then the automorphism induced on  $\sqcup_{i=0}^{m-1} \tilde{S}_i/B$  from the local monodromy (i.e. by an element of  $Sp(2g, \mathbb{Z}/m\mathbb{Z})$  fixing the point  $[B, b]$ ) in the sense of 4.6 permutes cyclically the  $m$  components of a semiabelian variety.*

*Proof.* Recall that the local monodromy about any point in a fine moduli space is finite. Since the monodromy is discrete we may assume that  $b, B$

are general. We argue by contradiction. Consider a morphism which fixes the components above, then on each component the automorphism lifts to an automorphism of the fiber bundle corresponding to  $b$  over  $B$ . Assume that an induced automorphism fixes the components, then it also sends the  $0$  and  $\infty$  sections of each component (which are the intersections of two components, and thus fixed) to themselves. For  $B$  general the group of finite order automorphisms of  $B$  is generated by the Kummer involution and translates by torsion points. However the torsion points of  $B$  are limits of torsion points coming from smooth abelian varieties, and the Kummer involution is the limit of Kummer involutions on smooth abelian varieties.

It remains to prove that if just one component is fixed then its  $0, \infty$  sections are fixed. We will assume the contrary and derive an intersection theoretic contradiction: assume that the  $0, \infty$  section are interchanged, then there exists an automorphism of the Chow group of the universal  $\mathbb{P}^1$  bundle over  $B$  (over some test curve  $C \subset B$ ) which permutes the  $0, \infty$  sections. However, the square of the infinity section is the first Chern class of the bundle, while the square of the  $0$  section is minus this first Chern class. As one of these is effective and one is anti-effective (as divisors on the  $0$  and  $\infty$  sections respectively), this is impossible.  $\square$

**4.8. Proposition-Definition.** There is a commutative diagram of finite quotients and inclusions:

$$\begin{array}{ccc} \mathcal{X}_g(m) & \xrightarrow{q} & \mathcal{X}_g \\ \downarrow & & \downarrow \\ \mathcal{X}_g(m)^1 & \xrightarrow{q^1} & \mathcal{X}_g^1. \end{array}$$

Fixing a moduli point in  $\mathcal{A}_g^1(m)$  and working over its pullback to  $\mathcal{X}_g^1(m)$ , the map  $q^1$  is defined on semiabelian varieties as a quotient by

- limits of translation by the universal  $m$  torsion points  $\{a_1, \dots, a_g\}$ , which are the  $m$  torsion point subgroup of  $B$ , and
- the component shift automorphism described in Lemma 4.7 above. We denote this component shift operator by  $S$ , for “shift”. Its relationship with the automorphism  $s$  of  $B \times B$  will be made clear in what follows.

and as  $q$  on abelian varieties.

*Proof.* By Lemma 4.7 our claim follows locally about a generic boundary point in  $\mathcal{X}^1(m)$ . However our definition of the automorphism group above is a global algebraic definition. Hence  $q^1$  is a quotient by a finite group of algebraic automorphism, which factors through the fibers of the map  $\mathcal{X}_g^1(m) \rightarrow \mathcal{A}_g^1(m)$ , as described locally in Lemma. Finally recall that the quotient of a flat family by a finite group which acts fiberwise is a flat family (we need the flatness in order to have finite group quotients of projective

varieties - flat families live in ambient projective spaces); hence the quotient by  $q^1$  is  $\mathcal{X}^1$ .  $\square$

**4.9. Remark.** One can alternatively describe the local picture of the action of  $\mathrm{Sp}(2g, \mathbb{Z}/m\mathbb{Z})$  on the universal family  $\mathcal{X}_g^1(m)$  near a boundary point of  $\mathcal{A}_g^1(m)$  analytically in coordinates, by studying the action of  $\mathrm{Sp}(2g, \mathbb{Z})$  on the universal cover  $\mathcal{H}_g \times \mathbb{C}^g$  (where  $\mathcal{H}_g$  stands for the Siegel upper half-space) of the universal family, similar to the discussion in [HuKaWe93]. In this case if the degeneration corresponds to  $ma_1$  (which is a period vector of the abelian variety) going to infinity, then the limits of points  $a_2, \dots, a_g, a'_2, \dots, a'_g$  are  $m$ -torsion points on  $B$  — and thus induce automorphisms of it by translation — while the limit of the translation by  $a_1$  is the shift  $S$ .

**4.10. Corollary.** *The pullback under  $q^1$  of the universal theta divisor in  $\mathcal{X}_g^1$  to  $\mathcal{X}_g^1(m)$  is a flat family.*

*Proof.* Both the pullback of the universal theta divisor under the finite group quotient  $q^1$ , and the universal  $m$ -theta divisor in  $\mathcal{X}_g^1(m)$  are flat families. Since they are equal on  $\mathcal{X}_g(m)$ , they are equal on  $\mathcal{X}_g^1(m)$ .  $\square$

**4.11. Corollary.** *Let  $[B, b]$  be a moduli point in  $\mathcal{A}_g^1$ , let  $F$  be a maximal isotropic group on  $B$ , considered as a “limit” of maximal isotropic groups over the open part in the sense of Proposition 4.8, and let  $\tilde{\mathcal{P}}'$  be the pullback of the Poincaré bundle on  $B/F \times B/F$  to  $B \times B$  under the quotient map  $B \times B \rightarrow (B/F) \times (B/F)$ . Then the restriction of the map  $q^1$  is the “clutching” (gluing the  $\infty$ -section of each  $\tilde{\mathcal{P}}'^{(i)}$  in some way to the 0-section of  $\tilde{\mathcal{P}}'^{(i+1)}$ , and the  $\infty$  of  $\tilde{\mathcal{P}}'^{(m-1)}$  to the 0 of  $\tilde{\mathcal{P}}'^{(0)}$  in some way) of the morphism given in the diagram:*

$$(\sqcup_{i=0}^{m-1} \tilde{\mathcal{P}}'^{(i)} / B \times B) / \underset{\text{Def. 4.4 on fibers}}{\text{identification of}} \rightarrow \tilde{\mathcal{P}}' / (B/F) \times (B/F) \\ (x_i, b_1, b_2) \mapsto (x, b_1 + F, b_2 + F).$$

*Specifically, the universal level abelian variety over  $[B]$  is the “clutching” of  $\sqcup_{i=0}^{m-1} \tilde{\mathcal{P}}'_i$  along the 0,  $\infty$  fibers.*

*Proof.* Let  $b \in B$  be a point in the “moduli”, or right, copy of  $B$ . The quotient of the fiber  $\tilde{S}(m)$  (corresponding to  $b$ ) over  $B \times \{b\} \subset B \times B$  by the group described in 4.8 is indeed the semiabelian variety corresponding to  $[B/F, b + F]$ . Hence our statement is true when restricted to  $B \times \{b\}$ .

This means that we have a map from any fiber of each  $\tilde{\mathcal{P}}'_i$  to the corresponding fiber of  $\tilde{\mathcal{P}}'$  defined naturally, and since a linear map of line bundles is a bundle morphism, the universal line bundle over  $B \times B$  is the pullback of the Poincaré bundle from  $(B/F) \times (B/F)$ .  $\square$

**4.12.** The above corollary describes the structure of the map  $q^1$  on the universal semiabelian family over  $B$ , and by working carefully with this description one can determine the appropriate gluings of the 0 and  $\infty$  sections of consecutive bundles, and to compute the classes of the bundles  $\tilde{\mathcal{P}}'_i$ .

However, for our computations it is much easier to change coordinates: indeed, we can translate each of  $B \times B$  by some point and pull back the bundles, before applying the map  $q^1$ . Note that shifting the origin on  $B \times B$  (i.e. apply a translation to it) changes the choice of the basis for  $NS_{\mathbb{Q}}(B \times B)$ , as the Poincaré bundle is not mapped to itself by a translation. The choice of translations that we make is as follows: it is the one induced by the component-shift action of  $S$  (which satisfies  $q^1 \circ S = q^1$ , this is true over smooth abelian varieties, and the limit is flat).

**4.13. Definition.** To define our new coordinates on each of  $\tilde{\mathcal{P}}_0^{(i)}$ , we do not apply a translation on the bundle  $\tilde{\mathcal{P}}_0^{(0)}$ , while for any  $i > 0$  we take the point  $S^i(0)$  to be the origin on  $\tilde{\mathcal{P}}_0^{(i)}$ . This corresponds to shifting coordinates on the base of each of the bundles. We choose a basis for each  $NS(\tilde{\mathcal{P}}_0^{(i)})$  in these coordinates as before, using  $\alpha'^{(i)} := c_1(\mathcal{P}'^{(i)})$  instead of  $\alpha$ . Since  $S(\mathcal{P}'^{(i)}) = \mathcal{P}'^{(i+1)}$ , in our new coordinates  $\alpha'^{(i)} = \alpha'^{(i+1)}$  (as classes in  $NS(B \times B)$ ), while if we did not shift the origins of each of  $\tilde{\mathcal{P}}_0^{(i)}$ , this would not be the case.

**The universal order  $m$  theta divisor on rank 1 degenerations.**

**4.14. Notation.** Given a class  $x \in NS(B \times B) = NS(\tilde{\mathcal{P}}_0^{(i)})$  we denote by  $\tilde{x}_i$  the class of its pullback to  $NS(\tilde{\mathcal{P}}_0^{(i)})$  (or simply  $\tilde{x}$ , if the choice of  $i$  is clear). We also denote by  $\xi_i \in NS(\tilde{\mathcal{P}}_0^{(i)})$  the first Chern class of the tautological line bundle on the  $\mathbb{P}^1$ -bundle  $\tilde{\mathcal{P}}_0^{(i)}$  over  $B \times B$ . As in section 2, we work with the image of  $r : NS(B \times B) \rightarrow NS(B \times C)$ .

**4.15.** Proposition 2.12 applies to each of the bundles  $\tilde{\mathcal{P}}_0^{(i)}$  to yield  $CH^*(\tilde{\mathcal{P}}_0^{(i)}) = CH^*(B \times B)[\xi_i]/(\xi_i^2 + \tilde{\alpha}'\xi_i = 0)$ . The discussion following Proposition 2.12 also applies to each of the  $\tilde{\mathcal{P}}_0^{(i)}$ , so that we get  $\mathcal{P}'_{\infty}^{(i)} = \xi_i$ , and  $\mathcal{P}'_0^{(i)} = \xi_i + \tilde{\alpha}'$ .

**4.16. Theorem.** *The class of the restriction of the universal theta divisor to  $\tilde{\mathcal{P}}_0^{(i)}$  in  $NS(\tilde{\mathcal{P}}_0^{(i)})$  is numerically equivalent to:*

$$\mathcal{D}_i \equiv \xi_i + m\tilde{\mu}_i + \frac{1}{2}\tilde{\alpha}'_i + \frac{m}{4}\tilde{\eta}_i$$

Moreover, the class  $\alpha'$  is equal to  $m\alpha$ , and thus we can write

$$\mathcal{D}_i \equiv \xi_i + m\tilde{\mu}_i + \frac{m}{2}\tilde{\alpha}_i + \frac{m}{4}\tilde{\eta}_i$$

*Proof.* The proof is very similar to the proof of Theorem 3.6, and is very easy in our coordinates. Indeed, since  $q^1 \circ S = q^1$ , we have  $S^*(D_{i+1}) = D_i$ . Since the coordinates are induced by the action of  $S$ , this means that in the expression

$$D_i = c_{\xi}^{(i)}\xi_i + c_{\mu}^{(i)}\tilde{\mu}_i + c_{\alpha}^{(i)}\tilde{\alpha}'_i + c_{\eta}^{(i)}\tilde{\eta}_i$$

all coefficients are independent of  $i$  (since for the basis we have  $S^*(\mu_{i+1}) = \mu_i$ , etc.).

We will thus compute these coefficients on the bundle  $\tilde{\mathcal{P}}^{(0)}$ , which is mapped to  $\tilde{\mathcal{P}}$  simply by the map  $q^1$ , with no shift involved. From Proposition 4.8 it follows that  $c_\xi = 1$ . By Corollary 4.11 we know that the base  $\tilde{\mathcal{P}}_0^{(i)} = B \times B$  is mapped to  $\tilde{\mathcal{P}}_0 = B/F \times B/F$  simply by taking the quotient, i.e. by the map  $q \times q$  for the  $g-1$ -dimensional abelian varieties.

To compute the pullbacks of the classes  $\mu, \alpha, \eta$  under the map  $q$ , note that the map  $q \times q$  induces the map  $q$  on the vertical and horizontal copies of  $B$  in  $B \times B$ , as well on the diagonal, and the test curves sit in these abelian varieties, they are all mapped to the image by the map  $q$ . Since the pullback of the theta divisor under  $q$  is a section of  $m\Theta$  (this is a classical theta function of order  $m$ ), this means that the divisor classes  $\mu, \alpha$ , and  $\eta$  pull back to  $m\mu, m\alpha$ , and  $m\eta$  respectively.

Moreover, the bundle  $\tilde{\mathcal{P}}^{(0)}$  is the pullback of  $\tilde{\mathcal{P}}$  under the map  $q$ , and thus we must have  $\alpha' = q^*\alpha$ , which by the above implies  $\alpha' = m\alpha$ .  $\square$

**4.17. Example.** As an example of using the machinery we will compute the boundary coefficient in  $NS(\mathcal{A}_g^1)$  of the branch divisor of  $f : m\Theta \hookrightarrow \mathcal{X}_g^1(m) \rightarrow \mathcal{A}_g^1(m) \rightarrow \mathcal{A}_g^1$ . Since  $\mathcal{A}_g(m)^1 \rightarrow \mathcal{A}_g^1$  is ramified to order  $m$  over the boundary, and since  $m\Theta$  is the pullback under the map  $q$  above of the divisor  $\Theta \subset \mathcal{X}_g^1$ , by restricting this picture to the test curve and computing in the universal family over it, we see that this coefficient is simply  $m^{g+1}$  times the coefficient computed in 3.9. To demonstrate our techniques we will compute this using the method of 3.9. This means we will take a test curve  $C$  in the boundary of  $\mathcal{A}_g^1$  contracted to a point  $B \in \mathcal{A}_{g-1}$  in the Satake compactification, and work in the universal level family over it. As before, we denote by  $n = C \cdot \Theta_B$ ; the branch divisor on a test curve is just a number of points, which we thus compute in terms of  $n$ .

Indeed, by the first part of the proof of 3.9 we have, substituting the expression for the class from Theorem 4.16:

$$\mathbb{R}(f) = \sum_{i=0}^{m-1} \mathcal{D}_i^{g+1} = \sum_{i=0}^{m-1} (\xi_i + m\tilde{\mu}_i + \frac{1}{2}\tilde{\alpha}'_i + \frac{m}{4}\tilde{\eta}_i)^{g+1}.$$

If we now denote  $\mu'_i := m\mu_i$  and  $\eta'_i := m\eta_i$  (and recall that  $\alpha'_i = m\alpha_i$ ), then all of the identities that were valid for  $\xi, \mu, \eta, \alpha$  in the no level case are valid for  $\xi_i, \mu'_i, \eta'_i, \alpha_i$  in the level case, except that now we get an extra factor of  $m^g$  for the top intersection numbers, i.e. we have

$$\begin{aligned} (\blacksquare') \quad & \xi_i^2 = -\tilde{\alpha}'_i \xi_i \\ (\square') \quad & (\eta'_i)^2 = 0 \quad (\diamondsuit') \quad (\eta'_i)(\mu'_i)^{g-1} = m^g n(g-1)! \\ (\triangle') \quad & \alpha'_i \eta'_i = 0 \quad (\heartsuit') \quad (\alpha'_i)^k (\mu'_i)^{g-k} = \begin{cases} -2m^g(g-2)! & \text{if } k=2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

It thus follows that if we use these classes in identity (T), there is simply an extra factor of  $m^g$ , and thus the computation in 3.9 carries over verbatim to yield

$$\mathbb{R}(f) = \sum_{i=0}^{m-1} \mathcal{D}_i^{g+1} = m \left( m^g \frac{n(g+1)!}{6} \right)$$

as expected.

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